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On a fourth order elliptic equation with supercritical exponent

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Abstract

This paper is concerned with the semi-linear elliptic problem involving nearly critical exponent (P_ε): $\Delta^2 u = |u|^{8/(n-4)+\varepsilon} u$ in Ω, $\Delta u = u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, and ε is a positive real parameter. We show that, for ε small, (P_ε) has no sign-changing solutions with low energy which blow up at exactly three points. Moreover, we prove that (P_ε) has no bubble-tower sign-changing solutions.

MSC: 35J20; 35J60

Keywords: nonlinear problem; critical exponent; sign-changing solutions; bubble-tower solution

1 Introduction and results

We consider the following semi-linear elliptic problem with supercritical nonlinearity:

$$(P_\varepsilon) \quad \begin{cases} \Delta^2 u = |u|^{p-1+\varepsilon} u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, ε is a positive real parameter and $p + 1 = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embedding of $H^2(\Omega) \cap H_0^1(\Omega)$ into $L^{p+1}(\Omega)$.

When the biharmonic operator in (P_ε) is replaced by the Laplacian operator, there are many works devoted to the study of the counterpart of (P_ε); see for example [1–6], and the references therein.

When $\varepsilon < 0$, many works have been devoted to the study of the solutions of (P_ε) see for example [7–9]. In the critical case, this problem is not compact, that is, when $\varepsilon = 0$ it corresponds exactly to the limiting case of the Sobolev embedding $H^2(\Omega) \cap H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, and thus we lose the compact embedding. In fact, van Der Vorst showed in [10] that (P_0) has no positive solutions if Ω is a starshaped domain. Whereas Ebobisse and Ould Ahmedou proved in [11] that (P_0) has a positive solution provided that some homology group of Ω is non-trivial. This topological condition is sufficient, but not necessary, as examples of contractible domains Ω on which a positive solution exists show [12].

In the supercritical case, $\varepsilon > 0$, the problem (P_ε) becomes more delicate since we lose the Sobolev embedding which is an important point to overcome. The problem (P_ε) was studied in [7] where the authors show that there is no one-bubble solution to the problem and there is a one-bubble solution to the slightly subcritical case under some suitable conditions. However, we proved in [13] that (P_ε) has no sign-changing solutions which blow up exactly at two points. In this work we will show the non-existence of sign-changing solutions of (P_ε) having three concentration points.

We note that problem (P_ε) has a variational structure. The related functional is

$$\inf J(u), \quad \text{where } J(u) := \frac{\int_\Omega |\Delta u|^2}{(\int_\Omega |u|^{p+1+\varepsilon})^{2/(p+1+\varepsilon)}}, u \in H^2(\Omega) \cap H_0^1(\Omega), u \not\equiv 0.$$

J satisfies the Palais-Smale condition in the subcritical case, while this condition fails in the critical case. Such a failure is due to the functions

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{(n-4)/2}}{(1+\lambda^2|x-a|^2)^{(n-4)/2}}, \quad c_0 = \big(n(n-4)\big(n^2-4\big)\big)^{(n-4)/8}, \lambda > 0, a \in \mathbb{R}^n. \quad (1.1)$$

c_0 is chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the following problem:

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n. \quad (1.2)$$

When we study problem (1.2) in a bounded smooth domain Ω, we need to introduce the function $P\delta_{(a,\lambda)}$ which is the projection of $\delta_{(a,\lambda)}$ on $H_0^1(\Omega)$. It satisfies

$$\Delta^2 P\delta_{(a,\lambda)} = \Delta^2 \delta_{(a,\lambda)} \quad \text{in } \Omega, \qquad \Delta P\delta_{(a,\lambda)} = P\delta_{(a,\lambda)} = 0 \quad \text{on } \partial\Omega.$$

These functions are almost positive solutions of (1.2).

We denote by G the Green's function defined by, $\forall x \in \Omega$,

$$\Delta^2 G(x,\cdot) = c_n \delta_x \quad \text{in } \Omega, \qquad \Delta G(x,\cdot) = G(x,\cdot) = 0 \quad \text{on } \partial\Omega,$$

where δ_x is the Dirac mass at x and $c_n = (n-4)(n-2)w_n$, with w_n is the area of the unit sphere of $\mathbb{R}^n$. We denote by H the regular part of G, that is,

$$H(x_1,x_2) = |x_1 - x_2|^{4-n} - G(x_1,x_2) \quad \text{for } (x_1,x_2) \in \Omega^2.$$

For $x = (x_1,x_2) \in \Omega^2 \setminus \Gamma$, with $\Gamma = \{(y,y) : y \in \Omega\}$, we denote by $M(x)$ the matrix defined by

$$M(x) = (m_{ij})_{1 \le i,j \le 2}, \quad \text{where } m_{ii} = H(x_i,x_i), m_{12} = m_{21} = G(x_1,x_2), \quad (1.3)$$

and let $\rho(x)$ be its least eigenvalue.

The space $H^2(\Omega) \cap H_0^1(\Omega)$ is equipped with the norm $\|\cdot\|$ and its corresponding inner product $\langle\cdot,\cdot\rangle$ defined by

$$\|u\| = \left(\int_\Omega |\Delta u|^2\right)^{1/2} \quad \text{and} \quad \langle u,v\rangle = \int_\Omega \Delta u \Delta v, \quad u,v \in H^2(\Omega) \cap H_0^1(\Omega). \quad (1.4)$$

Now, we are able to state our result.

Theorem 1.1 *Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 6$. If 0 is a regular value of $\rho(x)$, then there exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no sign-changing solutions u_ε which satisfy*

$$u_\varepsilon = P\delta_{(a_{\varepsilon,1},\lambda_{\varepsilon,1})} - P\delta_{(a_{\varepsilon,2},\lambda_{\varepsilon,2})} + P\delta_{(a_{\varepsilon,3},\lambda_{\varepsilon,3})} + v_\varepsilon, \tag{1.5}$$

with $|u_\varepsilon|_\infty^\varepsilon$ is bounded and

$$\begin{cases} a_{\varepsilon,i} \in \Omega, \quad \lambda_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to \infty \quad \text{for } i = 1,2,3, \\ \langle P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}, P\delta_{(a_{\varepsilon,j},\lambda_{\varepsilon,j})} \rangle \to 0 \quad \text{for } i \neq j \text{ and } \|v_\varepsilon\| \to 0 \text{ as } \varepsilon \to 0. \end{cases}$$

The second result deals with the phenomenon of bubble-tower solutions for the biharmonic problem (P_ε) with supercritical exponent. We will give a generalization of the result found in [13]. More precisely, we have the following.

Theorem 1.2 *Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$. There exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no solutions u_ε of the form*

$$u_\varepsilon = \sum_{i=1}^k \gamma_i P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})} + v_\varepsilon, \quad \text{with } \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \cdots \leq \lambda_{\varepsilon,k} \text{ and } |u_\varepsilon|_\infty^\varepsilon \text{ is bounded}, \tag{1.6}$$

where $k \geq 2$, $\gamma_i \in \{-1, 1\}$, $a_{\varepsilon,i} \in \Omega$, for each $i \leq j$, $\lambda_{\varepsilon,i}|a_{\varepsilon,i} - a_{\varepsilon,j}|$ is bounded and as $\varepsilon \to 0$, $\|v_\varepsilon\| \to 0$, $\lambda_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to +\infty$, $\langle P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}, P\delta_{(a_{\varepsilon,j},\lambda_{\varepsilon,j})} \rangle \to 0$ for $i \neq j$, and if $l \notin \{k-1, k\}$, $\lambda_{\varepsilon,l}|a_{\varepsilon,l} - a_{\varepsilon,l+1}| \to 0$, where $l = \min\{q : \gamma_q = \cdots = \gamma_k\}$.

The proof of our results will be by contradiction. Thus, throughout this paper we will assume that there exist solutions (u_ε) of (P_ε) which satisfy (1.5) or (1.6). In Section 2, we will obtain some information as regards such (u_ε) which allows us to develop Section 3 which deals with some useful estimates to the proof of our theorems. Finally, in Section 4, we combine these estimates to obtain a contradiction. Hence the proof of our results follows.

2 Preliminary results

In this section, we assume that there exist solutions (u_ε) of (P_ε) which satisfy

$$u_\varepsilon = \sum_{i=1}^k \gamma_i P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})} + v_\varepsilon, \tag{2.1}$$

with $|u_\varepsilon|_\infty^\varepsilon$ is bounded, $k \geq 2$, $a_{\varepsilon,i} \in \Omega$, and as $\varepsilon \to 0$, $\|v_\varepsilon\| \to 0$, $\lambda_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to +\infty$, $\langle P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}, P\delta_{(a_{\varepsilon,j},\lambda_{\varepsilon,j})} \rangle \to 0$ for $i \neq j$. Arguing as in [14] and [15], we see that for u_ε satisfying (2.1), there is a unique way to choose α_i, a_i, λ_i, and v such that

$$u_\varepsilon = \sum_{i=1}^k \gamma_i \alpha_i P\delta_{(a_i,\lambda_i)} + v, \tag{2.2}$$

$$\text{with } \begin{cases} \alpha_i \in \mathbb{R}, \qquad \alpha_i \to 1, \\ a_i \in \Omega, \qquad \lambda_i \in \mathbb{R}_+^*, \qquad \lambda_i d(a_i, \partial\Omega) \to +\infty, \\ v \to 0 \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega), \qquad v \in E, \end{cases} \tag{2.3}$$

where E denotes the subspace of $H_0^1(\Omega)$ defined by

$$E := \{w : \langle w, \varphi\rangle = 0, \forall \varphi \in \operatorname{Span}\{P\delta_i, \partial P\delta_i/\partial\lambda_i, \partial P\delta_i/\partial a_i^j, i \le k; j \le n\}\}. \tag{2.4}$$

Here, a_i^j denotes the jth component of a_i and in the sequel, in order to simplify the notations, we set $\delta_{(a_i,\lambda_i)} = \delta_i$ and $P\delta_{(a_i,\lambda_i)} = P\delta_i$. We always assume that u_ε (which satisfies (2.1)) is written as in (2.2) and (2.3) holds. From (2.1), it is easy to see that the following remark holds.

Lemma 2.1 [13] *Let u_ε satisfying the assumption of the theorems. λ_i occurring in* (2.2) *satisfies*

$$\lambda_i^\varepsilon \to 1 \quad \text{as } \varepsilon \to 0 \text{ for each } i \le k. \tag{2.5}$$

Remark 2.2 [2, 16] We recall the following estimate:

$$\delta_i^\varepsilon(x) - c_0^\varepsilon \lambda_i^{\varepsilon(n-4)/2} = O\big(\varepsilon \log\big(1 + \lambda_i^2|x - a_i|^2\big)\big) \quad \text{in } \Omega. \tag{2.6}$$

3 Some useful estimates

As usual in this type of problems, we first deal with the v-part of u_ε, in order to show that it is negligible with respect to the concentration phenomenon.

Lemma 3.1 *The function v defined in* (2.2), *satisfies the following estimate*:

$$\|v\| \le c\varepsilon + c\begin{cases} \sum_i \frac{1}{(\lambda_i d_i)^{n-4}} + \sum_{i\ne j} \varepsilon_{ij}(\log \varepsilon_{ij}^{-1})^{(n-4)/n} & \text{if } n < 12, \\ \sum_i \frac{1}{(\lambda_i d_i)^{(n+4)/2-\varepsilon(n-4)}} + \sum_{i\ne j} \varepsilon_{ij}^{(n+4)/2(n-4)}(\log \varepsilon_{ij}^{-1})^{(n+4)/2n} & \text{if } n \ge 12, \end{cases}$$

where $d_i := d(a_i, \partial\Omega)$ for $i \le k$ and for $i \ne j$, ε_{ij} is defined by

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{(4-n)/2}. \tag{3.1}$$

Proof The proof is the same as that of Lemma 3.1 of [13], so we omit it. □

Now, we state the crucial points in the proof of our theorems.

Proposition 3.2 *Assume that $n \ge 5$ and let α_i, a_i and λ_i be the variables defined in* (2.2) *with $k = 3$ and $\gamma_1 = -\gamma_2 = \gamma_3$. We have*

$$\begin{aligned} &\left|\alpha_i c_1 \frac{n-4}{2}\frac{H(a_i,a_i)}{\lambda_i^{n-4}} + \sum_{j\ne i}(-1)^{i+j}\alpha_j c_1\left(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{n-4}{2}\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}}\right) + \alpha_i\frac{n-4}{2}c_2\varepsilon\right| \\ &\quad \le c\varepsilon^2 + c\begin{cases} \sum_k \frac{1}{(\lambda_k d_k)^{n-2}} + \sum_{j\ne i}(\varepsilon_{ij}^{\frac{n}{n-4}}\log\varepsilon_{ij}^{-1} + \varepsilon_{ij}^2(\log\varepsilon_{12}^{-1})^{\frac{2(n-4)}{n}}) & \text{if } n \ge 6, \\ \sum_k \frac{1}{(\lambda_k d_k)^2} + \sum_{j\ne i}\varepsilon_{ij}^2(\log\varepsilon_{12}^{-1})^{2/5} & \text{if } n = 5, \end{cases} \end{aligned} \tag{3.2}$$

where $i, j \in \{1, 2, 3\}$ with $i \ne j$ and c_1, c_2 are positive constants.

Proof Let

$$c_1 = c_0^{\frac{2n}{n-4}} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+4)/2}}$$

and

$$c_2 = \frac{n-4}{2} c_0^{\frac{2n}{n-4}} \int_{\mathbb{R}^n} \log\bigl(1+|x|^2\bigr) \frac{|x|^2-1}{(1+|x|^2)^{n+1}}\, dx.$$

It suffices to prove the proposition for $i = 1$. Multiplying (P_ε) by $\lambda_1 \partial P\delta_1/\partial\lambda_1$ and integrating on Ω, we obtain

$$\begin{aligned} &\alpha_1 \int_\Omega \delta_1^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} - \alpha_2 \int_\Omega \delta_2^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} + \alpha_3 \int_\Omega \delta_3^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\quad = \int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1}. \end{aligned} \tag{3.3}$$

Using [17], we derive

$$\int_\Omega \delta_1^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} = \frac{n-4}{2} c_1 \frac{H(a_1,a_1)}{\lambda_1^{n-4}} + O\biggl(\frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^{n-1}}\biggr), \tag{3.4}$$

$$\int_\Omega \delta_j^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} = c_1 \biggl(\lambda_1 \frac{\partial\varepsilon_{1j}}{\partial\lambda_1} + \frac{n-4}{2} \frac{H(a_1,a_j)}{(\lambda_1\lambda_j)^{(n-4)/2}}\biggr) + R_j, \tag{3.5}$$

where $j = 2, 3$ and R_j satisfies

$$R_j = O\biggl(\sum_{k=1,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n-1}} + \varepsilon_{1j}^{\frac{n}{n-4}} \log \varepsilon_{1j}^{-1}\biggr). \tag{3.6}$$

For the other term of (3.3), we have

$$\begin{aligned} &\int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\quad = \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{p-1+\varepsilon} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3) \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\qquad + (p+\varepsilon) \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\qquad + O\biggl(\|v\|^2 + \sum_{i\neq j} \varepsilon_{ij}^{\frac{n}{n-4}} \log \varepsilon_{ij}^{-1}\biggr). \end{aligned} \tag{3.7}$$

Concerning the last integral, it can be written as

$$\begin{aligned} &\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\quad = \int_\Omega (\alpha_1 P\delta_1)^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} + O\biggl(\int_{\Omega\setminus A_j} P\delta_j^{p-1} P\delta_1 |v| + \int_{A_j} P\delta_1^{p-1} P\delta_2 |v|\biggr), \end{aligned} \tag{3.8}$$

where $A_j = \{x : 2\alpha_j P\delta_j \le \alpha_1 P\delta_1\}$ for $j = 2, 3$.

Observe that, for $n \geq 12$, we have $p - 1 = 8/(n-4) \leq 1$, thus

$$\int_{\Omega\setminus A_j} P\delta_j^{p-1} P\delta_1 |v| + \int_{A_j} P\delta_1^{p-1} P\delta_j |v| \leq c \int_{\Omega} |v| (\delta_1 \delta_j)^{\frac{n+4}{2(n-4)}} \\ \leq c \|v\| \varepsilon_{1j}^{(n+4)/2(n-4)} \big(\log \varepsilon_{1j}^{-1}\big)^{(n+4)/2n}. \tag{3.9}$$

But for $n < 12$, we have

$$\int_{\Omega\setminus A_j} P\delta_j^{p-1} P\delta_1 |v| + \int_{A} P\delta_1^{p-1} P\delta_j |v| \leq c \varepsilon_{1j} \big(\log \varepsilon_{1j}^{-1}\big)^{(n-4)/n} \|v\|. \tag{3.10}$$

For the other integral in (3.8), using [16, 17], and Remark 2.2, we get

$$\int_{\Omega} P\delta_1^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \\ = O\left(\|v\| \left[\varepsilon + \left(\frac{1}{(\lambda_1 d_1)^{\inf(n-4,(n+4)/2)}} \text{ (if } n \neq 12) + \frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^4} \text{ (if } n = 12) \right) \right] \right). \tag{3.11}$$

It remains to estimate the second integral of (3.7). We have

$$\int_{\Omega} |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{p-1+\varepsilon} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3) \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \\ = \int_{\Omega} (\alpha_1 P\delta_1)^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} - \int_{\Omega} (\alpha_2 P\delta_2)^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} + \int_{\Omega} (\alpha_3 P\delta_3)^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \\ - (p+\varepsilon) \left(\int_{\Omega} \alpha_2 P\delta_2 (\alpha_1 P\delta_1)^{p-1+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} - \int_{\Omega} \alpha_3 P\delta_3 (\alpha_1 P\delta_1)^{p-1+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \right) \\ + O\Big(\sum \varepsilon_{1j}^{\frac{n}{n-4}} \log \varepsilon_{1j}^{-1} \Big). \tag{3.12}$$

Now, using Remark 2.2 and [17], we have

$$\int_{\Omega} P\delta_1^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = \frac{n-4}{2} \left(c_2 \varepsilon + 2 c_1 \frac{H(a_1,a_1)}{\lambda_1^{n-4}} \right) \\ + O\left(\varepsilon^2 + \frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^{n-1}} + \frac{1}{(\lambda_1 d_1)^2} \text{ (if } n = 5) \right), \tag{3.13}$$

$$\int_{\Omega} P\delta_j^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = c_1 \left(\lambda_1 \frac{\partial \varepsilon_{1j}}{\partial \lambda_1} + \frac{n-4}{2} \frac{H(a_1,a_j)}{(\lambda_1 \lambda_j)^{(n-4)/2}} \right) + T_j, \tag{3.14}$$

$$p \int_{\Omega} P\delta_j P\delta_1^{p-1+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = c_1 \left(\lambda_1 \frac{\partial \varepsilon_{1j}}{\partial \lambda_1} + \frac{n-4}{2} \frac{H(a_1,a_j)}{(\lambda_1 \lambda_j)^{(n-4)/2}} \right) + T_j, \tag{3.15}$$

where for $i = 2, 3$,

$$T_i = O\Big(\varepsilon \varepsilon_{1j} \big(\log \varepsilon_{1j}^{-1}\big)^{\frac{n-4}{n}} \Big) + \left(\varepsilon_{1j}^{\frac{n}{n-4}} \big(\log \varepsilon_{1j}^{-1}\big) + \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^n} \text{ (if } n \geq 8) \right) \\ + \left(\frac{\varepsilon_{1j} (\log \varepsilon_{1j}^{-1})^{\frac{n-4}{n}}}{(\lambda_i d_i)^{n-4}} \text{ (if } n < 8) \right).$$

Therefore, combining (3.3)-(3.15), and Lemma 3.1, the proof of Proposition 3.2 follows. □

Proposition 3.3 *Let $n \geq 6$. We have the following estimate*:

$$\alpha_i \frac{1}{\lambda_i^{n-3}} \frac{\partial H(a_i, a_i)}{\partial a_i} - \frac{2}{\lambda_i} \sum_{j \neq i} (-1)^{i+j} \alpha_j \left(\frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{1}{(\lambda_i \lambda_j)^{(n-4)/2}} \frac{\partial H}{\partial a_i}(a_i, a_j) \right)$$
$$= O\left(\sum_k \frac{1}{(\lambda_k d_k)^{n-2}} + \sum_{j \neq i} \varepsilon_{ij}^{\frac{n}{n-4}} \log \varepsilon_{ij}^{-1} + \varepsilon_{ij}^2 \left(\log \varepsilon_{1j}^{-1}\right)^{\frac{2(n-4)}{n}} + \varepsilon^2 + \frac{\varepsilon}{(\lambda_i d_i)^{n-3}} \right),$$

where $i, j \in \{1, 2, 3\}$ and $j \neq i$.

Proof The proof is similar to the proof of Proposition 3.2. But there exist some integrals which have different estimates. We will focus in those integrals. In fact, (3.3), (3.7)-(3.12) are also true if we change $\lambda_1 \partial P\delta_1 / \partial \lambda_1$ by $(1/\lambda_1) \partial P\delta_1 / \partial a_1$. It remains to deal with the other equations. Following [17], we get

$$\int_\Omega \delta_1^p \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} = -\frac{1}{2} \frac{c_1}{\lambda_1^{n-3}} \frac{\partial H(a_1, a_1)}{\partial a_1} + O\left(\frac{1}{(\lambda_1 d_1)^{n-1}} \right), \tag{3.16}$$

$$\int_\Omega \delta_j^p \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} = \frac{c_1}{\lambda_1} \left(\frac{\partial \varepsilon_{1j}}{\partial a_1} - \frac{1}{(\lambda_1 \lambda_j)^{(n-4)/2}} \frac{\partial H}{\partial a_1}(a_1, a_j) \right)$$
$$+ O\left(\sum_{k=1,j} \frac{1}{(\lambda_k d_k)^{n-1}} + \lambda_j |a_1 - a_j| \varepsilon_{1j}^{(n-1)/(n-4)} \right), \tag{3.17}$$

$$\int_\Omega P\delta_1^{p+\varepsilon} \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} = -c_0^\varepsilon \lambda_1^{\varepsilon(n-4)/2} \frac{c_1}{\lambda_1^{n-3}} \frac{\partial H(a_1, a_1)}{\partial a_1} + O\left(\frac{1}{(\lambda_1 d_1)^{n-2}} + \frac{\varepsilon}{(\lambda_1 d_1)^{n-3}} \right), \tag{3.18}$$

$$\int_\Omega P\delta_j^{p+\varepsilon} \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial \lambda_1} = c_0^\varepsilon \lambda_j^{\varepsilon(n-4)/2} \left(P\delta_j, \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} \right) + O\left(\varepsilon \varepsilon_{1j} \left(\log \varepsilon_{1j}^{-1}\right)^{(n-4)/n} \right) + T_j, \tag{3.19}$$

$$\int_\Omega P\delta_j \frac{1}{\lambda_1} \frac{\partial (P\delta_1^{p+\varepsilon})}{\partial a_1} = c_0^\varepsilon \lambda_1^{\varepsilon(n-4)/2} \left(P\delta_j, \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} \right) + O\left(\varepsilon \varepsilon_{1j} \left(\log \varepsilon_{1j}^{-1}\right)^{(n-4)/n} \right) + T_j. \tag{3.20}$$

The proof of Proposition 3.3 is thereby completed. □

4 Proof of the theorems

Proof of Theorem 1.1

Arguing by contradiction, let us assume that problem (P_ε) has solutions (u_ε) as stated in Theorem 1.1. Recall that u_ε is written as

$$u_\varepsilon = \alpha_{\varepsilon,1} P\delta_{(a_{\varepsilon,1}, \lambda_{\varepsilon,1})} - \alpha_{\varepsilon,2} P\delta_{(a_{\varepsilon,2}, \lambda_{\varepsilon,2})} + \alpha_{\varepsilon,3} P\delta_{(a_{\varepsilon,3}, \lambda_{\varepsilon,3})} + v_\varepsilon,$$

with v_ε orthogonal to each $P\delta_{(a_i, \lambda_i)}$ and their derivatives with respect to λ_i and $(a_i)_k$, where $(a_i)_k$ denotes the kth component of a_i (see (2.2) and (2.3)). For simplicity, we will write $\alpha_i := \alpha_{\varepsilon,i}$, $\lambda_i := \lambda_{\varepsilon,i}$, and $a_i := a_{\varepsilon,i}$. From Proposition 3.2, for each $i = 1, 2, 3$, with $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = -1$. We have

$$(E_i) \quad c_1 \frac{n-4}{2} \frac{H(a_i, a_i)}{\lambda_i^{n-4}} + \gamma_i c_1 \sum_{j \neq i} \gamma_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-4)/2}} \right) + \frac{n-4}{2} c_2 \varepsilon$$
$$= o\left(\varepsilon + \sum_{j=1}^{3} \frac{1}{(\lambda_j d_j)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj} \right).$$

Furthermore, an easy computation shows that

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{n-4}{2}\varepsilon_{ij}\left(1 - 2\frac{\lambda_j}{\lambda_i}\varepsilon_{ij}^{2/n-4}\right) \quad \text{for } i,j = 1,2,3, j \neq i, \tag{4.1}$$

$$-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - 2\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geq \frac{n-4}{2}\varepsilon_{ij} \quad \text{for } \lambda_i \leq \lambda_j. \tag{4.2}$$

On the other hand, following the proof of Proposition 3.3, we have, for each $i = 1,2,3$,

$$(F_i) \quad \frac{1}{\lambda_i^{n-3}} \frac{\partial H(a_i,a_i)}{\partial a_i} - \sum_{j\neq i} 2\frac{(-1)^{j+i}}{\lambda_i}\left(\frac{\partial \varepsilon_{ji}}{\partial a_i} - \frac{\partial H(a_j,a_i)}{\partial a_i}\frac{1}{(\lambda_j\lambda_i)^{(n-4)/2}}\right)$$
$$= o\left(\sum_j \frac{1}{(\lambda_j d_j)^{n-3}} + \sum_{r\neq j} \varepsilon_{rj}^{\frac{n-3}{n-4}} + \varepsilon^{\frac{n-3}{n-4}}\right). \tag{4.3}$$

We distinguish many cases depending on the set

$$\mathcal{F} := \left\{(i,j) : i \neq j \text{ and } \min(\lambda_i,\lambda_j)|a_i - a_j| \text{ is bounded}\right\}$$

and we will prove that all these cases cannot occur.

We remark that if $(i,j) \in \mathcal{F}$ we derive $\lambda_i/\lambda_j \to 0$ or ∞ and $d_i/d_j = 1 + o(1)$ as $\varepsilon \to 0$.

Furthermore, the behavior of ε_{ij} depends on the set $\mathcal{F}$. In fact we have, assuming that $\lambda_i \leq \lambda_j$,

$$c\left(\frac{\lambda_i}{\lambda_j}\right)^{(n-4)/2} \leq \varepsilon_{ij} \leq \left(\frac{\lambda_i}{\lambda_j}\right)^{(n-4)/2} \quad \text{if } (i,j) \in \mathcal{F}, \tag{4.4}$$

$$\varepsilon_{ij} = \frac{1}{(\lambda_i\lambda_j|a_i - a_j|^2)^{(n-4)/2}} + o(\varepsilon_{ij}) \quad \text{if } (i,j) \notin \mathcal{F}. \tag{4.5}$$

First we start by proving the following crucial lemmas.

Remark 4.1 Ordering the λ_i's: $\lambda_{i_1} \leq \lambda_{i_2} \leq \lambda_{i_3}$, adding $(E_{i_1}) + 2(E_{i_2}) + 4(E_{i_3})$, and using (4.2), it is easy to derive a contradiction if we have $\varepsilon_{13} = o(\sum(\lambda_i d_i)^{4-n} + \sum \varepsilon_{rj} + \varepsilon)$.

Lemma 4.2 *Let $n \geq 4$. Then there exists a positive constant $\underline{c}_0 > 0$ such that*

(i) $\underline{c}_0^{-1} \leq \dfrac{d_1}{d_3} \leq \underline{c}_0$;

(ii) $\underline{c}_0^{-1} \leq \dfrac{\lambda_1}{\lambda_3} \leq \underline{c}_0$;

(iii) $\underline{c}_0^{-1} \leq \dfrac{|a_1 - a_3|}{d_i} \leq \underline{c}_0^{-1}$ *for $i = 1,3$.*

Proof The proof will be by contradiction.

Proof of (i). Assume that $d_1/d_3 \to 0$. In this case, we have

$$|a_1 - a_3| \geq cd_3 \quad \text{and} \quad \varepsilon_{13} = \frac{1}{(\lambda_1\lambda_3|a_1 - a_3|^2)^{(n-4)/2}} + o(\varepsilon_{13}), \tag{4.6}$$

which implies that $\varepsilon_{13} = o((\lambda_1 d_1)^{4-n} + (\lambda_3 d_3)^{4-n})$. Using Remark 4.1, we derive a contradiction. In the same way, we prove that $d_3/d_1 \nrightarrow 0$. Hence the proof of Claim (i) is completed.

Proof of (ii). Assume that $\lambda_1/\lambda_3 \to 0$. By Claim (i), we have $(\lambda_3 d_3)^{-1} = o((\lambda_1 d_1)^{-1})$. Four cases may occur.

Case 1. $\lambda_2/\lambda_3 \nrightarrow 0$ or $\{(1,2),(2,3)\} \cap F = \phi$. Using (4.5), (E_2) implies that

$$\frac{H(a_2,a_2)}{\lambda_2^{n-4}} + \varepsilon_{12} + \varepsilon_{23} + \varepsilon = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \varepsilon_{13}\right).$$

By Claim (i) and (E_3), we obtain $\varepsilon_{13} = o((\lambda_1 d_1)^{4-n})$. By Remark 4.1, this case cannot occur.

Case 2. $\lambda_2/\lambda_3 \to 0$, $\{(1,2),(2,3)\} \cap F \neq \phi$, and $\lambda_2/\lambda_1 \to +\infty$. In this case, it is easy to obtain $\varepsilon_{13} = o(\varepsilon_{12} + \varepsilon_{23})$. Using Remark 4.1, we derive a contradiction.

Case 3. $\lambda_2/\lambda_3 \to 0$, $(2,3) \in F$, $(1,2) \notin F$, and $\lambda_2/\lambda_1 \nrightarrow +\infty$. In this case, we see that $\lambda_2|a_2 - a_3|$ is bounded and $\lambda_2|a_1 - a_2| \to +\infty$. Hence, we derive that $\lambda_2|a_1 - a_3| \to +\infty$, which implies that $\lambda_k|a_1 - a_3| \to +\infty$ for $k = 1,3$. Thus

$$\varepsilon_{13} = \frac{1+o(1)}{(\lambda_1\lambda_3|a_1-a_3|^2)^{(n-4)/2}} = \left(\frac{\lambda_2}{\lambda_3}\right)^{(n-4)/2} \frac{1+o(1)}{(\lambda_1\lambda_2|a_1-a_3|^2)^{(n-4)/2}} = o(\varepsilon_{23}).$$

Then by Remark 4.1, we get a contradiction.

Case 4. $\lambda_2/\lambda_3 \to 0$, $(1,2) \in F$, and $\lambda_2/\lambda_1 \nrightarrow +\infty$. In this case, it is easy to get $\varepsilon_{23} = o(\varepsilon_{12})$.

Using the formula $[(E_1) + (E_2) - (E_3)]$, we deduce that $\varepsilon = o(\varepsilon_{12} + \varepsilon_{13})$, which implies that $\varepsilon_{13} = o(\varepsilon_{12})$. Hence by Remark 4.1, we derive a contradiction and Claim (ii) is thereby completed.

Proof of (iii). Without loss of generality, we can assume that $d_1 \leq d_3$. First, as in the proof of Claim (i), we get $|a_1 - a_3| \leq c_0 d_1$. Now assume that $|a_1 - a_3|/d_1 \to 0$, which implies

$$\frac{H(a_i,a_i)}{\lambda_i^{n-4}} = o(\varepsilon_{13}) \quad \text{for } i = 1,3.$$

Two cases may occur.

Case 1. $\lambda_1 \leq \lambda_2$ or $\{(1,2),(2,3)\} \cap F = \phi$. Using (E_2), we obtain

$$\frac{H(a_2,a_2)}{\lambda_2^{n-4}} = o(\varepsilon_{13}), \qquad \varepsilon_{i2} = o(\varepsilon_{13}) \quad \text{for } i = 1,3 \quad \text{and} \quad \varepsilon = o(\varepsilon_{13}),$$

and we derive a contradiction from (E_1).

Case 2. $\lambda_2 \leq \lambda_1$ and $\{(1,2),(2,3)\} \cap F \neq \phi$. Let $k \in \{1,3\}$ such that $(2,k) \in F$. Using Claim (ii) and the fact that $\lambda_2 \leq \lambda_1$, we derive that $\varepsilon_{2k} \geq c(\lambda_2/\lambda_k)^{(n-4)/2}$, which implies that $d_2 \sim d_k$, $\lambda_2/\lambda_k \to 0$, and $\lambda_2|a_2 - a_k|$ is bounded. Using (4.3) for $i = k$, we get

$$-\lambda_2|a_2 - a_k|\varepsilon_{2k}^{\frac{n}{n-4}} + \frac{\lambda_1\lambda_3}{\lambda_k}|a_1 - a_3|\varepsilon_{13}^{\frac{n}{n-4}} = o\left(\frac{1}{(\lambda_2 d_2)^{n-3}} + \sum_{r\neq j} \varepsilon_{rj}^{\frac{n-3}{n-4}} + \varepsilon^{\frac{n-3}{n-4}}\right). \tag{4.7}$$

Since $\lambda_2|a_2 - a_k|$ is bounded and $\varepsilon_{13} \simeq (\lambda_1\lambda_3|a_2 - a_k|^2)^{(4-n)/2}$, we derive that

$$\varepsilon_{13}^{\frac{n-3}{n-4}} = o\left(\frac{1}{(\lambda_2 d_2)^{n-3}} + \varepsilon_{12}^{\frac{n-3}{n-4}} + \varepsilon_{23}^{\frac{n-3}{n-4}} + \varepsilon^{\frac{n-3}{n-4}}\right),$$

which implies that

$$\varepsilon_{13} = o\left(\frac{1}{(\lambda_2 d_2)^{n-4}} + \varepsilon_{12} + \varepsilon_{23} + \varepsilon\right). \tag{4.8}$$

By Remark 4.1, we get a contradiction. □

Lemma 4.3 *There exists a positive constant* $\underline{c}'_0$ *such that*

(i) $\underline{c}'_0 \lambda_1 \leq \lambda_2$;

(ii) $d_i \geq \underline{c}'_0$ *for* $i = 1, 3$.

Proof Without loss of generality, we can assume that $d_1 \leq d_3$.

Proof of (i). Assume that $\lambda_2/\lambda_1 \to 0$. First we claim that $d_1/d_2 \nrightarrow 0$. In fact, arguing by contradiction we assume that $d_1/d_2 \to 0$, we get $d_1 \to 0$, $|a_1 - a_2| \geq cd_2$, and $|a_2 - a_3| \geq cd_2$. Hence, $\{(1,2),(2,3)\} \cap \digamma = \phi$. From (E_2), we obtain

$$\frac{H(a_2,a_2)}{\lambda_2^{n-4}} + \varepsilon_{12} + \varepsilon_{23} + \varepsilon = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \frac{1}{(\lambda_3 d_3)^{n-4}} + \varepsilon_{13}\right). \tag{4.9}$$

Let ν_i be the outward normal vector at a_i. Since d_1, d_3, and $|a_1 - a_3|$ are of the same order, we have (see [18] and [19])

$$\frac{1}{\lambda_1^{n-3}} \frac{\partial H(a_1,a_1)}{\partial \nu_1} \sim \frac{c}{(\lambda_1 d_1)^{n-3}} \quad \text{and} \quad \frac{\partial G(a_1,a_3)}{\partial \nu_1} \leq 0. \tag{4.10}$$

Using (F_1), we get $1/(\lambda_1 d_1)^{n-3} = o(\varepsilon_{13}^{(n-3)/(n-4)})$, which implies that $1/(\lambda_1 d_1)^{n-4} = o(\varepsilon_{13})$. From (E_1), we derive a contradiction. Hence our claim is proved.

Thus there exists a positive constant c so that $d_1 \geq cd_2$. Now, since we have assumed that $\lambda_2/\lambda_1 \to 0$, Lemma 4.2 implies that $\varepsilon_{13} = o((\lambda_2 d_2)^{4-n})$. Finally, using Remark 4.1, we get a contradiction and the proof of Claim (i) follows.

Proof of (ii). Assume that $d_1 \to 0$. Note that Claim (i) and (E_2) imply that (4.9) holds.

Now, following the proof of (i), we obtain a contradiction. □

We turn now to the proof of Theorem 1.1. By the previous lemmas, we know that λ_1 and λ_3 are of the same order, $|a_1 - a_3| \geq c$ and $\lambda_2 \geq c\lambda_i$, for $i = 1, 3$ where c is a positive constant.

Hence, (E_2) implies that (4.9) holds. Furthermore, for $i = 1, 3$ (E_i) implies that

$$\frac{H(a_i,a_i)}{\lambda_i^{n-4}} - \frac{G(a_1,a_3)}{(\lambda_1 \lambda_3)^{n-4}} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \frac{1}{(\lambda_3 d_3)^{n-4}} + \varepsilon_{13}\right). \tag{4.11}$$

We denote by $r(x)$ the eigenvector associated to $\rho(x)$ whose norm is 1. We point out that we can choose $r(x)$ so that all their components are positive (see [18] and [19]).

Let $\Lambda_i = \lambda_i^{(4-n)/2}$, $\Lambda = (\Lambda_1, \Lambda_3)$, and $x = (a_1, a_3)$. From (4.11), we have

$$M(x) \cdot \frac{{}^t\Lambda}{\|\Lambda\|} = o(1). \tag{4.12}$$

The scalar product of (4.12) by $r(x)$ gives

$$\rho(x)r(x)\cdot\frac{{}^t\Lambda}{\|\Lambda\|}=o(1). \tag{4.13}$$

Since the components of $r(x)$ are positive and λ_1, λ_3 are of the same order, there exists a positive constant c, such that $r(x)\cdot\frac{{}^t\Lambda}{\|\Lambda\|}\geq c>0$. Hence, we get

$$\rho(x)=o(1). \tag{4.14}$$

We deduce from (4.3) and (4.11) that

$$\frac{\partial M}{\partial x_i}(x)\cdot\frac{{}^t\Lambda}{\|\Lambda\|}=o(1). \tag{4.15}$$

Observe that Λ may be written in the form

$$\Lambda=\beta r(x)+\overline{r}(x),\quad \text{with } r(x)\cdot\overline{r}(x)=0, \|\overline{r}\|=o(\beta) \text{ and } \beta\sim\|\Lambda\|. \tag{4.16}$$

Using (4.15), we get

$$\frac{\partial M}{\partial x_i}(x)\cdot{}^t r(x)+\frac{\partial M}{\partial x_i}(x)\cdot\frac{\overline{r}(x)}{\|\Lambda\|}=o(1). \tag{4.17}$$

Since $d_i\geq c_0$ for $i=1,3$ and $|a_1-a_3|\geq c_0$, the matrix $\frac{\partial M}{\partial x_i}(x)$ is bounded.

Furthermore, we have $\|\overline{r}\|=o(\|\Lambda\|)$, which implies that

$$\frac{\partial M}{\partial x_i}(x)\cdot{}^t r(x)=o(1). \tag{4.18}$$

Let us consider the equality

$$M(x)\cdot{}^t r(x)=\rho(x)\cdot{}^t r(x)$$

and derivative it with respect to x_i; we obtain

$$\frac{\partial M}{\partial x_i}(x)\cdot{}^t r(x)+M(x)\frac{\partial^t r}{\partial x_i}(x)=\frac{\partial\rho}{\partial x_i}(x)\cdot{}^t r(x)+\rho(x)\frac{\partial^t r}{\partial x_i}(x).$$

The scalar product with $r(x)$ gives

$$r(x)\cdot\frac{\partial M}{\partial x_i}(x)\cdot{}^t r(x)=\frac{\partial\rho}{\partial x_i}(x). \tag{4.19}$$

Using (4.18), we obtain

$$\frac{\partial\rho}{\partial x_i}(x)=o(1). \tag{4.20}$$

Hence, we derive a contradiction from (4.14), (4.20), and the fact that 0 is a regular value of ρ. Thus the proof of our theorem follows.

Proof of Theorem 1.2

Arguing by contradiction, let us assume that problem (P_ε) has solutions (u_ε) as stated in Theorem 1.2. From Section 2, these solutions have to satisfy (2.2) and (2.3).

As in the proof of Proposition 3.2, we have, for each $i = 1, \dots, k$,

$$(E_i) \quad c_1 \frac{n-4}{2} \frac{H(a_i,a_i)}{\lambda_i^{n-4}} + \gamma_i c_1 \sum_{j \neq i} \gamma_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2} \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}} \right) + \frac{n-4}{2} c_2 \varepsilon = o\left(\varepsilon + \sum_{j=1}^{k} \frac{1}{(\lambda_j d_j)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj} \right).$$

Observe that, if $j < i$, we have $\lambda_j |a_i - a_j|$ is bounded (by the assumption) which implies that

$$\begin{aligned} &|a_i - a_j| = o(d_j), \qquad d_i/d_j = 1 + o(1), \quad \forall i,j \quad \text{and} \\ &\varepsilon_{ij} \geq c(\lambda_j/\lambda_i)^{(n-4)/2}, \quad \forall j < i, \end{aligned} \tag{4.21}$$

where c is a positive constant. Using (4.21), easy computations show that

$$\begin{aligned} &\varepsilon_{(i-1)j} + \varepsilon_{i(j+1)} = o(\varepsilon_{ij}), \quad \forall i < j, \\ &\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} \right) \quad \text{if } (i,j) \neq (1,1). \end{aligned} \tag{4.22}$$

Thus, using (4.22), (E_i) can be written as

$$\begin{aligned} &(E_1') \quad c_1 \frac{n-4}{2} \frac{H(a_1,a_1)}{\lambda_1^{n-4}} + c_1\gamma_1\gamma_2\lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + \frac{n-4}{2} c_2 \varepsilon = o\left(\varepsilon + \frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj} \right), \\ &(E_k') \quad c_1\gamma_{k-1}\gamma_k\lambda_k \frac{\partial \varepsilon_{(k-1)k}}{\partial \lambda_k} + \frac{n-4}{2} c_2 \varepsilon = o\left(\varepsilon + \frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj} \right), \end{aligned}$$

and for $1 < i < k$,

$$(E_i') \quad c_1\gamma_{i-1}\gamma_i\lambda_i \frac{\partial \varepsilon_{(i-1)i}}{\partial \lambda_i} + c_1\gamma_i\gamma_{i+1}\lambda_i \frac{\partial \varepsilon_{i(i+1)}}{\partial \lambda_i} + \frac{n-4}{2} c_2 \varepsilon = o\left(\varepsilon + \frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj} \right).$$

The proof will depend on the value of l which is defined in the theorem.

Case 1. $l = k$. From the definition of l we get $\gamma_{k-1}\gamma_k = -1$. Now using (4.1) and (E_k'), we derive that

$$\varepsilon = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r \neq j} \varepsilon_{ij} \right) \quad \text{and} \quad \varepsilon_{(k-1)k} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj} \right). \tag{4.23}$$

Now, using (4.23) and (E_{k-1}'), we derive the estimate of $\varepsilon_{(k-2)(k-1)}$ and by induction we get

$$\varepsilon_{(i-1)i} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj} \right) \quad \text{for each } i = 2, \dots, k. \tag{4.24}$$

Finally, using (4.22), (4.23), (4.24), and (E_1') we obtain

$$\frac{H(a_1,a_1)}{\lambda_1^{n-4}} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}}\right),$$

which gives a contradiction.

Case 2. $l = k-1$. Using (4.1), an easy computation implies that

$$\lambda_{k-1}\frac{\partial\varepsilon_{(k-1)k}}{\partial\lambda_{k-1}} - \lambda_k\frac{\partial\varepsilon_{(k-1)k}}{\partial\lambda_k} \geq c\varepsilon_{(k-1)k}. \tag{4.25}$$

Then from (E_{k-1}'), (E_k'), (4.1), (4.25), and the fact that $\gamma_{k-1}\gamma_k = 1$ and $\gamma_{k-2}\gamma_{k-1} = -1$ (since $l = k-1$), we obtain

$$c\varepsilon_{(k-1)k} + \varepsilon_{(k-2)(k-1)} = o\left(\varepsilon + \frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r\neq j}\varepsilon_{rj}\right). \tag{4.26}$$

Now using (E_k') and (4.26) we get (4.23) and as before, (4.24) is satisfied. Hence we also derive a contradiction from (E_1').

Case 3. $l \notin \{k, k-1\}$. Recall that in this case we have assumed that $\lambda_l|a_l - a_{l+1}| \to 0$. This implies that

$$\lambda_l\frac{\partial\varepsilon_{l(l+1)}}{\partial\lambda_l} = \big((n-4)/2\big)\varepsilon_{l(l+1)}\big(1+o(1)\big). \tag{4.27}$$

Hence, using (E_l'), the definition of l and (4.1) we obtain the first part of (4.23). The second part follows from (E_k') and the first one. Finally, as before we derive a contradiction from (E_1').

Hence, our theorem is proved.

Competing interests
The author declares that they have no competing interests.

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